

We may consider the parametric equations of torus:

$$x = (c + r \cdot \cos v) \cdot \cos u$$

$$y = (c + r \cdot \cos v) \cdot \sin u$$

$$z = r \cdot \sin v$$

For an effective calculation, the Parametric system is more convenient than the Cartesian system.

For a segment of torus shell, corresponding to an X axis that is within the bisecting plane, the following limits are valid:

$$r \in [a-t, a]$$

$$v \in [0, 2\pi]$$

$$u \in [-\alpha, \alpha]$$

For clarifications, please consider the figure. Don't forget that  $\alpha$  must be in radian units!

The geometric centroid  $x_G$  coordinate is:

$$x_G = \frac{\iiint_V x dV}{\iiint_V dV} \quad (\text{and note that, choosing Cartesian coordinate system as in figure, } y_G \text{ and } z_G \text{ are vanishing})$$

The elementary volume is given by  $dV = dx \cdot dy \cdot dz = J \cdot dr \cdot dv \cdot du$ , where  $J$  is Jacobian determinant of coordinates transform (from Cartesian system to Parametric system)  $(x, y, z) \rightarrow (r, v, u)$ , i.e.  $J = r \cdot (c + r \cdot \cos v)$  [you may consider <http://mathworld.wolfram.com/Torus.html> for this results; or you may calculate it- it's not so difficult...]

The coordinate  $x_G$  is given by formula:

$$x_G = \frac{\iiint_V x dV}{\iiint_V dV} = \frac{I_1}{I_2} = \frac{\int_{a-t}^a \int_{-\alpha}^{+\alpha} \int_0^{2\pi} r \cdot (c + r \cdot \cos v)^2 \cdot \cos u \cdot du \cdot dv \cdot dr}{\int_{a-t}^a \int_{-\alpha}^{+\alpha} \int_0^{2\pi} r \cdot (c + r \cdot \cos v) \cdot du \cdot dv \cdot dr}$$

The Triple-Integrals evaluation is below:

$$\begin{aligned} I_1 &= \int_{-\alpha}^{+\alpha} \cos u \cdot du \cdot \int_0^{2\pi} \int_{a-t}^a [rc^2 + r^3 \cos^2 v + 2cr^2 \cos v] \cdot dr \cdot dv = 2 \sin \alpha \int_0^{2\pi} \left[ c^2 \frac{r^2}{2} + \cos^2 v \cdot \frac{r^4}{4} + 2c \cos v \cdot \frac{r^3}{3} \right]_{r=a-t}^{r=a} dv = \\ &= 2 \sin \alpha \left[ c^2 \frac{a^2 - (a-t)^2}{2} v \Big|_{v=0}^{v=2\pi} + \frac{a^4 - (a-t)^4}{4} \left( \frac{v}{2} + \frac{\sin 2v}{4} \right) \Big|_{v=0}^{v=2\pi} + 2c \frac{a^3 - (a-t)^3}{3} \sin v \Big|_{v=0}^{v=2\pi} \right] = \\ &= 2 \sin \alpha \left[ c^2 \frac{a^2 - (a-t)^2}{2} 2\pi + \frac{a^4 - (a-t)^4}{4} \pi \right] = 2\pi \sin \alpha \left[ c^2 (a^2 - (a-t)^2) + \frac{a^4 - (a-t)^4}{4} \right] \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{-\alpha}^{+\alpha} du \cdot \int_0^{2\pi} \int_{a-t}^a [rc + r^2 \cos v] \cdot dr \cdot dv = 2\alpha \int_0^{2\pi} \left[ c \frac{r^2}{2} + \cos v \cdot \frac{r^3}{3} \right]_{r=a-t}^{r=a} dv = \\ &= 2\alpha \left[ c \frac{a^2 - (a-t)^2}{2} v \Big|_{v=0}^{v=2\pi} + \frac{a^3 - (a-t)^3}{3} \sin v \Big|_{v=0}^{v=2\pi} \right] = 2\alpha \left[ c \frac{a^2 - (a-t)^2}{2} 2\pi \right] = 2\pi \alpha \cdot c (a^2 - (a-t)^2) \end{aligned}$$

The final result is:

$$\begin{aligned} x_G &= \frac{\iiint_V x dV}{\iiint_V dV} = \frac{I_1}{I_2} = \frac{\sin \alpha}{\alpha} \frac{c^2 (a^2 - (a-t)^2) + \frac{a^4 - (a-t)^4}{4}}{c (a^2 - (a-t)^2)} = \frac{\sin \alpha}{\alpha} \left( c + \frac{a^4 - (a-t)^4}{4c (a^2 - (a-t)^2)} \right) = \\ &= \frac{\sin \alpha}{\alpha} \left( c + \frac{(a^2 - (a-t)^2) \cdot (a^2 + (a-t)^2)}{4c (a^2 - (a-t)^2)} \right) = \frac{\sin \alpha}{\alpha} \left( c + \frac{a^2 + (a-t)^2}{4c} \right) \end{aligned}$$

where

$c$  is the radius of elbow,  $2\alpha$  is the angle of elbow, in radians (e.g.  $\alpha = \pi/4$  for a 90 degree elbow),  $a$  is the external radius of section (i.e. a half of OD of pipe) and  $t$  is the thickness of elbow.

