Solving the fluid equations

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Airflow contributes to the convective effect in forced and natural convection. Air is assumed to be an incompressible Newtonian fluid that can be modeled by the following form of the Navier-Stokes equation:

$$\frac{\partial \mathbf{v}}{\partial t} = \alpha \nabla^2 \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p + \mathbf{f} \qquad (1)$$

where **v** is the velocity vector, α is the kinematic viscosity, p is the pressure, and **f** is the body force such as gravity or buoyancy. The first term on the right-hand side describes the diffusion of momentum. The second term describes advection. The mass continuity condition ensures the conservation of mass:

$$\nabla \cdot \mathbf{v} = 0 \qquad (2)$$

For the temperature distribution to affect the flow of the air, we can add a thermal buoyancy term to the body force $\mathbf{f}(x, y, z, t)$:

$$\mathbf{f}(x, y, z, t) = \gamma \left[T(x, y, z, t) - T_0(x, y, t) \right] \vec{\mathbf{z}} \qquad (3)$$

where $T_0(x, y, t)$ is the average temperature of the column of air at the location (x, y) and at time t and γ is the thermal expansion coefficient.

The Navier-Stokes equation

Eq. (1) can be decomposed into two steps (assuming a uniform distribution of pressure or incorporating the pressure gradient term into the body force term):

1. The diffusion step

$$\frac{\partial \mathbf{v}_1}{\partial t} = \alpha \nabla^2 \mathbf{v}_1 + \mathbf{f} \qquad (4)$$

2. The advection step

$$\frac{\partial \mathbf{v}_2}{\partial t} = -(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2 \qquad (5)$$

The diffusion step Eq. (4) can be solved efficiently using a relaxation method like in the case of solving the heat equation. The advection step Eq. (5) can be solved using the MacCormack method[1], which consists of the following two steps:

1. The predictor step

$$\bar{\xi}^{n+1}_{i,j,k} = \xi^n_{i,j,k} - \frac{\Delta t}{2\Delta x}\left[u^n_{i+1,j,k}\xi^n_{i+1,j,k} - u^n_{i-1,j,k}\xi^n_{i-1,j,k}\right] - \frac{\Delta t}{2\Delta y}\left[v^n_{i,j+1,k}\xi^n_{i,j+1,k} - v^n_{i,j-1,k}\xi^n_{i,j-1,k}\right]$$
$$- \frac{\Delta t}{2\Delta z}\left[w^n_{i,j,k+1}\xi^n_{i,j,k+1} - w^n_{i,j,k-1}\xi^n_{i,j,k-1}\right] \qquad (6)$$

2. The corrector step

$$\xi^{n+1}_{i,j,k} = \frac{1}{2}\left[\xi^n_{i,j,k} + \bar{\xi}^{n+1}_{i,j,k}\right] - \frac{\Delta t}{4\Delta x}u^n_{i,j,k}\left[\bar{\xi}^{n+1}_{i+1,j,k} - \bar{\xi}^{n+1}_{i-1,j,k}\right] - \frac{\Delta t}{4\Delta y}v^n_{i,j,k}\left[\bar{\xi}^{n+1}_{i,j+1,k} - \bar{\xi}^{n+1}_{i,j-1,k}\right]$$
$$- \frac{\Delta t}{4\Delta z}w^n_{i,j,k}\left[\bar{\xi}^{n+1}_{i,j,k+1} - \bar{\xi}^{n+1}_{i,j,k-1}\right] \qquad (7)$$

The function ξ can be any component of the velocity (u, v, w).

The continuity condition

The solution of Eq. (1) will not automatically satisfy the conservation of mass. The mass continuity condition Eq. (2) must be imposed at each step. This condition is also important in simulating convective loops as it forces the velocity fields to have vortices. The discretized form of the continuity equation requires that:

$$\nabla \cdot \mathbf{V} = \frac{u^n_{i+1,j,k} - u^n_{i-1,j,k}}{2\Delta x} + \frac{v^n_{i,j+1,k} - v^n_{i,j-1,k}}{2\Delta y} + \frac{w^n_{i,j,k+1} - w^n_{i,j,k-1}}{2\Delta z} = 0 \qquad (8)$$

Recall **the Helmholtz decompostion**[2] that states any vector field can be decomposed into the following form:

$$\mathbf{V} = \mathbf{V}' + \nabla\phi \qquad (9)$$

where **V'** is a divergence-free vector field ($\nabla \cdot \mathbf{V}' = 0$) and ϕ is a scalar field. We can apply the gradient operator to both sides of Eq. (9) and obtain:

$$\nabla \cdot \mathbf{V} = \nabla^2\phi \qquad (10)$$

This is a **Poisson equation** for the scalar field ϕ, given that **V** is known. If ϕ can be solved, then we can fix the solution by subtracting $\nabla\phi$ from **V**[3]:

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[1] http://en.wikipedia.org/wiki/MacCormack_method

[2] http://en.wikipedia.org/wiki/Helmholtz_decomposition

$$\mathbf{V}' = \mathbf{V} - \nabla\phi \qquad (11)$$

The discretized form of this equation is simply:

$$u^n_{i,j,k} := u^n_{i,j,k} - \frac{\phi^n_{i+1,j,k} - \phi^n_{i-1,j,k}}{2\Delta x} \qquad (12a)$$

$$v^n_{i,j,k} := v^n_{i,j,k} - \frac{\phi^n_{i,j+1,k} - \phi^n_{i,j-1,k}}{2\Delta y} \qquad (12b)$$

$$w^n_{i,j,k} := w^n_{i,j,k} - \frac{\phi^n_{i,j,k+1} - \phi^n_{i,j,k-1}}{2\Delta z} \qquad (12c)$$

Again, the Poisson equation can be solved by using a relaxation method[4]. The method uses the following iteration until convergence is reached:

$$\phi^n_{i,j,k} := \frac{1}{\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2} + \frac{2}{(\Delta z)^2}} \left\{ \frac{\phi^n_{i+1,j,k} + \phi^n_{i-1,j,k}}{(\Delta x)^2} + \frac{\phi^n_{i,j+1,k} + \phi^n_{i,j-1,k}}{(\Delta y)^2} + \frac{\phi^n_{i,j,k+1} + \phi^n_{i,j,k-1}}{(\Delta z)^2} - \nabla \cdot \mathbf{V} \right\} \qquad (13)$$

A multigrid method[5] may be used to accelerate the convergence. If accuracy is less important than visual effect, 5-20 iteration steps can be used without worrying about if convergence has been reached.

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[3] N. Foster and D. Metaxas, *Modeling the Motion of a Hot, Turbulent Gas*, Proceedings of the 24th annual conference on Computer graphics and interactive techniques, Pages: 181 - 188, 1997

[4] http://en.wikipedia.org/wiki/Relaxation_method

[5] http://en.wikipedia.org/wiki/Multigrid_method